

A brief summary of the mathematics of a Rubik's cube

Mei Hammond-Tse

1 Introduction

The seemingly innocent Rubik's cube, invented by Ernő Rubik in 1974, has caused a ripple in popular culture with the rise of competitions and world championships as well as variations on the original 3 by 3 cube. What started as an inventive way to demonstrate particular architectural challenges developed into a puzzle which proved almost impossible to solve. It took a month for Rubik to solve his cube. However, as of 11th of June 2023, the world record for a single Rubik's cube solve is 3.13 seconds by Max Park. But what makes it so difficult to solve? The answer may be in the fact there are around 43 quintillion possibilities for how the cube can be scrambled. And this is just the solvable scrambled. It can also include how any move can cause any number of problems by messing up other areas of the cube which have previously been solved. I'm sure, if you have ever attempted to solve a Rubik's cube, you know the nightmare of when you try to solve another side but end up destroying your previous work.

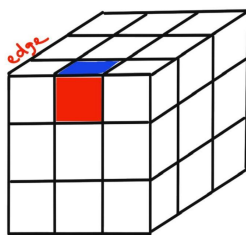
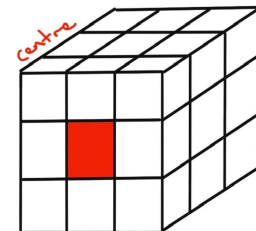
The key to solving our cube involves some pretty complicated maths and theories which seem pretty convoluted for such a seemingly simple toy. Here I am going to provide an outline of some of the fascinating maths behind the Rubik's cube.

2 Components of a Rubik's Cube

To first understand where the colossal 43 quintillion comes from, we must first think of the structure of our Rubik's cube. Being a cube, it has 6 faces which are each a different colour. Each face is made up of a 3 x 3 grid of smaller squares or 'cubies'. This results in a total of 26 coloured pieces and 54 smaller coloured faces from the cubies on the whole cube. The smaller cubies and internal core and screws mean each of the faces can rotate clockwise or anti-clockwise in increments of 90 degrees.

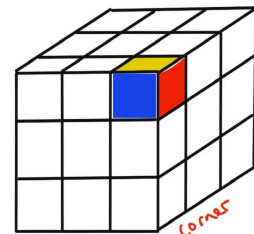
There are three different types of cubie on the Rubik's cube:

Centres are only one colour and are attached to the internal core of the puzzle. You could think of cores as being stationary or fixed. This is because each colour of centre is always in the same place in relation to the other centres. So, in a conventional cube, yellow is always opposite white, green is always opposite blue and red is always opposite orange.



The next type of cubie is an edge. These have 2 colours and 'connect' the centre pieces together. They can have two possible orientations and one edge can go in one of 12 possible positions on the cube.

Finally, a corner cubie has three colours and they make up the corners of the cube. They can be flipped in three possible ways and any one corner can be in one of the eight corners of the whole cube.



3 Why 43 quintillion?

The actual number of possible solvable combinations is 43,252,003,274,489,856,000. This comes about from considering how each of the different types of cubies can be oriented in the cube.

Let us begin with the edges. There are 12 edges which can be oriented in 2 different ways so the permutation (or the number of ways the edges can be arranged) of edges as 2^{12} arrangements. The 12 pieces can be in one of 12 different positions on the cube so, the number of arrangements is $(12 \cdot 11 \cdot 10 \cdot \dots \cdot 1)$ or $12!$ (read as 12 factorial, the notation for the product of descending numbers up to 1).

The orientations of the corners are next. There are 8 edges which can each be oriented in 3 ways so the permutation of corners is 3^8 arrangements. The 8 different pieces can be in 8 different positions on the cube so, the number of arrangements is $(8 \cdot 7 \cdot 6 \cdot \dots \cdot 1)$ or $8!$.

This results in $(12! \cdot 2^{12}) \cdot (8! \cdot 3^8) = 5.19024039 \cdot 10^{20}$ total possible combinations of how the cube can be arranged. However, there are many combinations which are unsolvable. It is proven that $(12! \cdot 2^{12}) \cdot (8! \cdot 3^8)$ can be divided by 12 in order to eliminate any of the unsolvable combinations. But where does 12 come from? If we break down our 12 into its prime factors, $(2 \cdot 2 \cdot 3)$, we can deal with each one separately.

The first 2 stems from the fact that only $\frac{1}{2}$ of the permutations of the edges will be correct. Also, the factor of 3 comes about from a similar concept of only $\frac{1}{3}$ of the permutations of the corners will be correct.

The last factor of 2 is a special case which involves the parity (whether something is odd or even) of the cube. Any series of movements for the permutation of the cube can be rewritten in pairs of which coloured face of a cubie has been moved through a series of moves. If any one of the faces is rotated 90° in any direction, an even number of cubies will be displaced (eg. if the right face is rotated once in the clockwise direction, the four corner cubies are moved and so are the four edge cubies on the face). This means it can be said that any number of rotations of a face, n , the number of cubies to be displaced is an even number (or multiple of 4 which is an even number). This means its parity is always even, even if n is odd.

What all this means in essence is that, given that the parity of a rubik's cube is 2, a single cubie cannot be displaced or relocated on the cube by itself without changing any other cubie on the cube. So only half of the permutations of the rubik's cube can have the correct orientation.

Tying this all together, the number of possible solvable combinations of a rubik's cube is worked out like this:

$$\frac{(12! \cdot 2^{12}) \cdot (8! \cdot 3^8)}{2 \cdot 2 \cdot 3} = 43,252,003,274,489,856,000$$

4 Group Theory

The key to the solution of the Rubik's cube lies in a mathematical theory known as Group Theory.

Definition group, G - a set of binary objects and a binary operator, $*$, which satisfies the four requirements or axioms.

Axiom 1 - G is closed. This means that when the operator $*$ is applied to two objects (eg. h and g) in G , the product of $h * g$ is also in the set of G .

Axiom 2 - the operator $*$ is associative. This means that, for any elements in the set (eg. h , g and f) the operation $h * (g * f) = (h * g) * f$. Simplified, the brackets around the elements in the operation can be moved and changed and the output would be the same. This is applicable to the rubik's cube as, as long as the order of rotations is the same, it does not matter how the moves are grouped.

Axiom 3 - there is an identity element $e \in G$ where $e * g = g * e = g$. For the Rubik's cube, the identity element is 0.

Axiom 4 - Every element in G has an inverse of g^{-1} . This also means that $g * g^{-1} = g^{-1} * g = e$. This is applicable to the Rubik's cube as a rotation in one direction can be rotated in the opposite direction by the same amount and it would equal 0.

4.1 Group theory theorems

There are some basic theorems or rules about groups. They are:

- The identity element e is unique (there are no other elements in the group which are equivalent to it)
- If $a * b = e$ then $a = b^{-1}$. This means that if a and b have the operator $*$ performed on them and the product is e then the two must be inverses of each other.
- If $a * x = b * x$ then $a = b$. Essentially, this means that if the product of a and a constant x is equal to the product of b and the same constant x , then a and b must be equal.
- The inverse of $(a * b)$ is $(b^{-1} * a^{-1})$. This is true because, if we think of the Rubik's cube, the inverse of any move a is turning the face in the opposite direction so is a^{-1} . The two operations are in the opposite order as, to reverse the sequence, the inverse operations must be completed in the opposite order.
- $(a^{-1})^{-1} = a$. This is like if there is a double negative, they cancel each other out. For our Rubik's cube, if you do the inverse of an inverse rotation of the face, the resulting movement is the permutation is the same as it was initially.

4.2 Group Theory and the Rubik's cube

Group Theory is used to model the Rubik's cube as each of the permutations or arrangements of the cube can be thought of as objects in the group of the Rubik's cube which can be notated as R .

Our binary operator which we are denoting as $*$ can be thought of as the connector between two moves of the cube's faces. The rotation of any of the faces in any orientation will equal another permutation of the cube. Hence, the group R is closed and can be thought of as a group.

As outlined previously, the inverse of a movement of one of the faces of a Rubik's cube is rotating the face the same amount in the opposite direction.

5 God's Number

Another interesting mathematical aspect of the Rubik's cube is God's number. This is the term coined for the minimum number of moves it takes to solve any scramble of the Rubik's cube puzzle.

5.1 How God's Number was found

Mathematicians managed to progressively come closer and closer to the value until, in 2010, the value was experimentally found by a group of mathematicians using 35 CPU-years worth of computing power donated to them by Google. They did this by solving every single possible combination of the cube. They divided the possible scrambles into 2,217,093,120 sets of 19,508,428,800 each. Then, due to the symmetry of the cube, the number of sets was reduced to 55,882,296. This is because the cube can be oriented 24 different ways in space and another factor of 2 if the cube was mirrored. This means the 2,217,093,120 sets take into account all orientations. However, the solution for each of the 48 orientations of the same scramble would be the same so they could eliminate the other 47 scrambles in the same set. The mathematicians then used a computer program they made to solve each of the scrambles in the sets they formed in less than 20 moves. They found that the minimum number of moves it takes to solve any scramble was 20 moves.

6 Conclusion

There is such a myriad of depth within the seemingly simple yet complex Rubik's cube. I have only scratched the surface of the mathematics behind it and the theorems which can be more easily visualised using it. So, next time you see a Rubik's cube, remember it's not all it first appears.

7 Bibliography

Ferenc, D. (2010). *Mathematics of the Rubik's Cube - Permutation Group*. [online] Ruwix.com.

Available at:

<https://ruwix.com/the-rubiks-cube/mathematics-of-the-rubiks-cube-permutation-group/> [Accessed 29 Mar. 2024].

Linkletter, D. (2021). *To Solve the Rubik's Cube, You Have to Understand the Amazing Math Inside*.

[online] Popular Mechanics. Available at:

<https://www.popularmechanics.com/science/math/a30244043/solve-rubiks-cube/> [Accessed 29 Mar. 2024].

mit.edu. (2009). *The Mathematics of the Rubik's Cube: Introduction to Group Theory and*

Permutation Puzzles. [online] Available at: <https://web.mit.edu/sp.268/www/rubik.pdf> [Accessed 29 Mar. 2024].

Rokicki, T., Kociemba, H., Davidson, M. and Dethridge, J. (2005). *God's Number is 20*. [online]

Cube20.org. Available at: <https://cube20.org/> [Accessed 29 Mar. 2024].

Wikipedia Contributors (2019). *Ernő Rubik*. [online] Wikipedia. Available at:

https://en.wikipedia.org/wiki/Ern%C5%91_Rubik [Accessed 29 Mar. 2024].

Wikipedia. (2024). *Speedcubing*. [online] Available at:

<https://en.wikipedia.org/wiki/Speedcubing#:~:text=As%20of%20July%202023%2C%20the> [Accessed 29 Mar. 2024].

www.youtube.com. (2018). *Group Theory and the Rubik's Cube*. [online] Available at:

<https://www.youtube.com/watch?v=D1iEAaHT1P8> [Accessed 29 Mar. 2024].